Algebra I: Goup theory
End semestral exam (online)

Time: 2 hours 30 minutes
Total score: 50

Notations: Throughout, G shall denote a group. S_n denotes the symmetric group of degree n, D_{2n} denotes the dihedral group of order $2n$.

1. State true or false. Justify your answers. No marks will be awarded in absence of proper justification.
   (a) G cannot be isomorphic to a proper quotient of itself.
   (b) There exists a non-abelian group of order 25.
   (c) $Z(S_n) = \{1\}$ for all $n \geq 3$.
   (d) If every subgroup of a group G is normal, then G is abelian.
   (e) Every infinite group has at least one element of infinite order. (5×2)

2. Determine the automorphism group of S_3. (8)

3. State and prove Cauchy's theorem for a finite group G. (8)

4. Let G be a group of order 203. Prove that if H is a normal subgroup of order 7 in G then $H \subseteq Z(G)$. Deduce that G is abelian in this case. (8)

5. Show that the order of the centralizer $C_{S_n}((12)(34))$ is $(n-4)! \times 8$ for all $n \geq 4$. Determine the elements of the centralizer explicitly. (8)

6. Show that a Sylow p-subgroup of the dihedral group D_{2n} is cyclic and normal for every odd prime p. (8)

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